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STUDY BY THE PRANDTL-GLAUERT METHOD OF COMPRESSIBILITY EFFECTS AND CRITICAL MACH NUMBER FOR ELLIPSOIDS OF VARIOUS ASPECT RATIOS AND THICKNESS RATIOS

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SUMMARY

By the use of a form of the Prandtl-Glauert method that is valid for three-dimensional flow problems, the value of the maximum incremental velocity for compressible flow about thin ellipsoids at zero angle of attack is calculated as a function of the Mach number for various aspect ratios and thickness ratios. The critical Mach numbers (within the accuracy of the Prandtl-Glauert method) of the various ellipsoids are also determined. The results indicate an increase in critical Mach number with decrease in aspect ratio which is large enough to explain experimental results on low-aspect-ratio wings at zero lift.

INTRODUCTION

Recent tests (references 1 and 2) have shown that an appreciable increase in the critical Mach number, together with other improvements of the aerodynamic characteristics at supercritical Mach numbers, results from the use of wings of very low aspect ratio. These improved characteristics have been somewhat qualitatively ascribed to "three-dimensional relief," although no quantitative theoretical discussion has yet been provided.

In the present paper an effort is made to provide such a study by considering the flow, at zero angle of attack, about a series of thin ellipsoids of various aspect ratios and thickness ratios. Ellipsoids were chosen because they are amenable to calculation. Although they differ appreciably from the wings of reference 1, which had an NACA 0012 airfoil section and rectangular plan form, ellipsoids should nevertheless show similar aspect-ratio effects. The calculations were made for ellipsoids of thickness ratios 0.10, 0.15, and 0.20, and for the entire range of aspect ratios from the elliptic cylinder to the ellipsoid of revolution.

The compressibility effects were computed by the use of a form of the Prandtl-Glauert method that is valid for three-dimensional flow

problems. The method has been given by Göthert (reference 3) without, however, very explicit mathematical proof. Another correct statement of the three-dimensional form of the Prandtl-Glauert method was given earlier by A. Busemann in reference 4, where, however, no formulas were given. Since the methods that have been commonly used (see, for example, references 5, 6, and 7) are applicable only to two-dimensional problems, a detailed proof of the method correct for three-dimensional flow is included in the appendix. A brief discussion of the accuracy of the Prandtl-Glauert method, as applied to ellipsoids, is also given. This work was completed in April 1946.

SYMBOLS

U	free-stream velocity
C	velocity of sound in free stream
M	free-stream Mach number (U/C)
γ	ratio of specific heats $(\gamma = 1.4$ for air)
$\beta = \sqrt{1 - M^2}$	
x, y, z	rectangular coordinates
B	thin body
φ	velocity potential
u, v, w	x-, y-, and z-components of incremental velocity for compressible flow about B
B'	body obtained by stretching B in direction of x-axis by the factor $1/\beta$
u', v', w'	x-, y-, and z-components of incremental velocity for incompressible flow about B'
a	maximum semichord of ellipsoid
b	semispan of ellipsoid
c	maximum semithickness of ellipsoid
$a' = \dfrac{a}{\beta}$	

A	aspect ratio $\left(A = \frac{(2b)^2}{\pi ab} = \frac{4}{\pi}\frac{b}{a}\right)$
$\bar{u} = \frac{u_{max}}{U}$	
$\bar{u}(M)$	value of $\bar{u}$ when the Mach number is equal to M
$\bar{u}(0)$	value of $\bar{u}$ for incompressible flow $(M = 0)$
ϵ	thickness ratio $\left(\frac{\text{Thickness}}{\text{Chord}}\right)$
$\frac{u}{U}(\epsilon,M)$	value of ratio of incremental velocity to free-stream velocity for compressible flow having Mach number M about a body having thickness ratio ϵ
$\frac{u}{U}(\epsilon,0)$	value of ratio of incremental velocity to free-stream velocity for incompressible flow about a body having thickness ratio ϵ

Subscript:

max	maximum value

METHODS OF CALCULATION

The Prandtl-Glauert method for three-dimensional flow.- The Prandtl-Glauert method is used in the present paper in the following form:

The incremental velocities at a point P of a three-dimensional compressible flow field about a thin body B may be obtained in three steps:

(1) The x-coordinates of all points of B are increased by the factor $1/\beta$, where

$$\beta = \sqrt{1 - M^2}$$

and where the x-axis is in the stream direction. This transformation takes B into a stretched body B'.

(2) The incremental velocities u', v', w', in the direction of the x-, y-, and z-axes, respectively, at the point P' in the flow field of B' corresponding to the point P in the flow field of B are calculated as though B' were in an incompressible flow having the same free-stream velocity as the original compressible flow.

(3) The values u, v, and w of the incremental velocities at the point P in the compressible flow field of the original unstretched body are then found by the equations

$$u = \frac{1}{\beta^2} u'$$

$$v = \frac{1}{\beta} v'$$

$$w = \frac{1}{\beta} w'$$

A derivation of this form of the Prandtl-Glauert method is given in the appendix. The method in essentially this form has been given by Göthert (reference 3) without, however, a very clear proof. (Göthert prefers to shrink the lateral coordinates of the body by the factor β rather than to expand the coordinate in the stream direction by the factor $1/\beta$; obviously the two procedures lead to the same result.) Prandtl (reference 5) and Von Kármán (reference 6) state the method in a form that is valid for two-dimensional flows but in general is incorrect for three-dimensional flows. Goldstein and Young (reference 7) also give a discussion leading to results that are correct only for two dimensions. A discussion of the reasons for the failure of these commonly used methods for three-dimensional flow problems is included in the appendix.

Calculation of incremental velocity for compressible flow about ellipsoids.- In order to determine, by the Prandtl-Glauert method, the incremental velocity on the surface of an ellipsoid having semiaxes a, b, and c (where a is the length of the semiaxis in the stream direction), the incremental velocity is calculated for a stretched ellipsoid having semiaxes a', b, and c (where $a' = \frac{a}{\beta}$ in an incompressible flow having the same stream velocity) and the result is multiplied by $1/\beta^2$. For incompressible flow about the stretched ellipsoid, the velocity potential on the surface of the ellipsoid is given by

$$\varphi = \frac{\alpha_o}{2 - \alpha_o} Ux$$

where

$$\alpha_0 = a'bc \int_0^{\infty} \frac{d\lambda}{\left(a'^2 + \lambda\right) \sqrt{\left(a'^2 + \lambda\right)\left(b^2 + \lambda\right)\left(c^2 + \lambda\right)}}$$

(See, for example, reference 8.) The incremental velocity at $x = 0$ (half-chord line on the stretched ellipsoid in incompressible flow) is then given by

$$u' = \frac{\alpha_0}{2 - \alpha_0} U$$

This value is the maximum value of u' (reference 8) and evidently is the same at all points on the half-chord line. The incremental velocity at the half-chord line for the compressible flow about the original unstretched ellipsoid is given by

$$u = \frac{1}{\beta^2} u' = \frac{1}{\beta^2} \frac{\alpha_0}{2 - \alpha_0} U \tag{1}$$

Various formulas are necessary for the evaluation of the integral α_0 when $a' > b > c$, $b > a' > c$, or $a' > b = c$ (ellipsoid of revolution).

For $a' > b > c$, the value of α_0 is given by the formula

$$\alpha_0 = \frac{2a'bc}{\left(a'^2 - b^2\right) \sqrt{a'^2 - c^2}} (F - E) \tag{2}$$

where F and E are incomplete elliptic integrals of the first and second kind, respectively, defined as follows:

$$F = \int_0^{\varphi} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}}$$

$$E = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \psi}\, d\psi$$

where

$$k = \sqrt{\frac{a'^2 - b^2}{a'^2 - c^2}}$$

and

$$\sin \varphi = \frac{\sqrt{a'^2 - c^2}}{a'}$$

For $b > a' > c$, the value of α_o is given by the formula

$$\alpha_o = \frac{2a'bc\sqrt{b^2 - c^2}}{\left(b^2 - a'^2\right)\left(a'^2 - c^2\right)}\left[E - \left(\frac{a'^2 - c^2}{b^2 - c^2}\right)F\right] - \frac{2c^2}{a'^2 - c^2} \tag{3}$$

where F and E are defined by the foregoing formulas but

$$k = \sqrt{\frac{b^2 - a'^2}{b^2 - c^2}}$$

and

$$\sin \varphi = \frac{\sqrt{b^2 - c^2}}{b}$$

Equation (2) is derived from the first equation given in equations (5.13) of reference 8 by substituting a' for a and by using the expression for k in terms of a', b, and c. Equation (3) is derived from the second equation given in equations (5.13) of reference by interchanging a and b, substituting a' for a, and using the expression for k in terms of a', b, and c.

For $a' > b = c$ (ellipsoid of revolution), α_o is given by the equation

$$\alpha_o = a'b^2 \int_0^\infty \frac{d\lambda}{\left(a'^2 + \lambda^2\right)^{3/2}\left(b^2 + \lambda\right)}$$

which resolves into

$$\alpha_o = \frac{1 - e^2}{e^3} \left(\log_e \frac{1 + e}{1 - e} - 2e \right)$$

where

$$e = \frac{\sqrt{a'^2 - b^2}}{a'}$$

If this value for α_o is substituted in equation (1), the incremental velocity at the half-chord line for the ellipsoid of revolution is found to be

$$u = \frac{1}{\beta^2} \frac{\log_e \frac{1 + e}{1 - e} - 2e}{\frac{2e}{1 - e^2} - \log_e \frac{1 + e}{1 - e}} U$$

The limiting case of infinite aspect ratio (elliptic cylinder) was treated by the use of formulas for the ellipse in two-dimensional flow (reference 9).

Calculation of the critical Mach number.- For flow about a two-dimensional body, the free-stream Mach number for which sonic speed is first reached at some point on the surface is called the critical Mach number, because of the development of a shock and the accompanying deterioration of the aerodynamic characteristics shortly after this Mach number is exceeded. If an extension of the definition of the critical Mach number to the general three-dimensional body is desired, the definition appears, at first sight, to require some revision, since for the general three-dimensional case the shock formation on parts of the body may occur along lines yawed with respect to the free-stream velocity (reference 10). The boundary lines of the supersonic regions (sonic lines) must, however, always contain a portion that is normal to the free-stream velocity and thus the definition of the critical Mach number for the two-dimensional case (infinite unyawed cylinder) may be extended without modification to the general three-dimensional case. For the special three-dimensional case of the infinite yawed cylinder, the portions of the boundary lines of the supersonic regions that are normal to the free-stream velocity are represented by two points at infinity, only one of which (at the downstream end) has the necessary qualities for accumulating disturbances, that is, for shock formation. It can be seen that for the general three-dimensional body the critical Mach number, although still defined in the same manner as

for the two-dimensional body, largely loses its critical significance, since the shock may begin to form over only a very small part of the surface, so that its occurrence does not necessarily imply an imminent deterioration of the aerodynamic characteristics of the body.

For the special case of the unyawed ellipsoid considered in the present paper, however, no appreciable analysis of shock formation or shock extent along the lines just indicated seems to be required. As is shown in the section "Calculation of the Incremental Velocity for Compressible Flow about Ellipsoids", the maximum velocity for an unyawed ellipsoid is in the stream direction and occurs simultaneously at all points along the half-chord line. Sonic velocity is thus reached simultaneously along a line that extends across the entire span of the body and is normal to the stream direction. These conditions also exist in the case of the unyawed infinite cylinder, that is, the two-dimensional body.

The critical Mach number of the ellipsoid, within the accuracy of the Prandtl-Glauert method, was accordingly determined by solving graphically the equation

$$\bar{u}(M) = \frac{1}{M}\sqrt{\frac{1 + \frac{\gamma - 1}{2} M^2}{\frac{\gamma + 1}{2}}} - 1$$

where $\bar{u}(M)$ is the ratio of the incremental velocity at the half-chord line to the stream velocity at the Mach number M.

<u>Accuracy of the Prandtl-Glauert method</u>.- The Prandtl-Glauert method is based on the assumption of small perturbations. Consequently, near the nose of the ellipsoids discussed in the present paper, where the assumption of small perturbations is violated, the results given by the Prandtl-Glauert method cannot be expected to be reliable. More reliable values, however, should be obtained for the maximum incremental velocity, which occurs at the half-chord line. The accuracy of the Prandtl-Glauert approximation for the maximum incremental velocity may be estimated by comparison with more exact solutions of the compressible flow problem. An iteration method in which the Prandtl-Glauert method is used as the first approximation has been proposed by Busemann (reference 11). The first and second approximations have been calculated by Hantzsche and Wendt for the elliptic cylinder (reference 12) and by Schmieden and Kawalki for the ellipsoid of revolution (reference 13). Calculation of the maximum incremental velocity for the elliptic cylinder having thickness ratio 0.20 by a formula for the second approximation given in reference 11 shows that the value given by the Prandtl-Glauert method

at a Mach number of 0.8 is almost 20 percent lower than the value given by the second approximation. For the ellipsoid of revolution, however, the value of the maximum incremental velocity given by the Prandtl-Glauert method agreed with the value given by the second approximation to within 5 percent at a Mach number of 0.8 for thickness ratios up to 0.30. Although the second approximation is not the exact solution, it indicates that the error involved in using the Prandtl-Glauert method to estimate the maximum incremental velocity for ellipsoids having a given thickness ratio is greatest for the limiting case of the elliptic cylinder $(A = \infty)$ and very small for the ellipsoid of revolution, which has a very low aspect ratio. The error may be expected to be intermediate in magnitude for intermediate values of the aspect ratio and to decrease with aspect ratio. The reduction of error of the Prandtl-Glauert method with a decrease in aspect ratio was to be expected, as the incremental velocities are smaller for ellipsoids having low aspect ratio.

RESULTS AND DISCUSSION

Results.- Figures 1 to 3 show the value of the velocity ratio $\bar{u} = \frac{u_{max}}{U}$ at the half-chord line plotted against the Mach number for ellipsoids at zero angle of attack for various aspect ratios and section thickness ratios equal to 0.10, 0.15, and 0.20. In the same figures the sonic velocity boundary having the equation

$$\bar{u} = \frac{1}{M}\sqrt{\frac{1 + \frac{\gamma - 1}{2} M^2}{\frac{\gamma + 1}{2}}} - 1$$

is plotted for air. The abscissa of the intersection of this boundary line with the curve of $\bar{u}$ plotted against M for any aspect ratio is the critical Mach number (within the accuracy of the Prandtl-Glauert method). In order to show the effect of compressibility more directly, the ratio $\frac{\bar{u}(M)}{\bar{u}(0)}$ of maximum incremental velocity for compressible flow to the maximum incremental velocity for incompressible flow for the same free-stream velocity is plotted against the Mach number in figures 4 to 6 for the same aspect ratios and thickness ratios. Similar curves for the ellipsoid of revolution, which is a special case of the ellipsoid having three unequal axes, are plotted for the same thickness ratios in figures 1 to 6. Figure 7 presents curves of critical Mach number against aspect ratio for thickness ratios of 0.10, 0.15, and 0.20.

Three-dimensional relief.- It may be seen from figures 1 to 3 that the three-dimensional relief, that is, the difference between the velocity

on the ellipsoid and the velocity on the corresponding ellipsoid of infinite aspect ratio (elliptic cylinder), increases with a decrease in the aspect ratio. This increase has two causes:

(1) For a flow at small values of M (incompressible flow), the relief effect increases with a decrease in the aspect ratio.

(2) For larger values of M (compressible flow), an additional relief effect occurs with a decrease in the aspect ratio because of the fact that the compressibility effect (increase of incremental velocity with an increase in the Mach number) decreases with a decrease in the aspect ratio. (See figs. 4 to 6.) It may be seen that this additional three-dimensional relief increases most rapidly at high Mach numbers.

Figures 1 to 6 show that the compressibility effect on the maximum incremental velocity is greatest for A equal to infinity (infinite elliptic cylinder) and is smallest for the ellipsoid of revolution. The compressibility effect on the maximum incremental velocity for the elliptic cylinder is proportional to $\frac{1}{\sqrt{1 - M^2}}$, which is in agreement with the usual form of the Prandtl-Glauert method in two dimensions. The compressibility effect on the maximum incremental velocity for the ellipsoid of revolution is small in comparison with that of the elliptic cylinder. In fact, as the thickness ratio of any type of body of revolution approaches zero, the compressibility correction factor approaches unity, for in this limit the incremental velocity in incompressible flow is proportional to the square of the thickness ratio, so that the effect of stretching the body (first step of Prandtl-Glauert method, see appendix) is exactly compensated for by the multiplication of the incremental velocities by $1/\beta^2$ (third step of the Prandtl-Glauert method). For ellipsoids of practical thickness ratios, however, the incremental velocity varies more slowly than the square of the thickness ratio. The compressibility effect for the ellipsoid of revolution (figs. 4 to 6) is thus appreciable at high Mach numbers. For example, for a thickness ratio of 0.20 and at a Mach number of 0.8, the compressibility effect amounts to about 30 percent of the incremental velocity in incompressible flow.

The effect of the thickness ratio on the three-dimensional relief may be seen by a comparison of figures 1, 2, and 3. From figure 1 it may be seen that, for a thickness ratio of 0.10, at a Mach number of 0.75, the maximum incremental velocity for $A = 2$ is 76 percent of the maximum incremental velocity for $A = \infty$. From figure 3, on the other hand, it may be seen that, for a thickness ratio of 0.20, at a Mach number of 0.75, the maximum incremental velocity for $A = 2$ is 75 percent of the maximum incremental velocity for $A = \infty$. Thus, an increase in the thickness ratio causes only a very small increase in the three-dimensional relief.

Critical Mach number. Figures 1, 2, 3, and 7 indicate that an increase in the critical Mach number of an ellipsoid at zero lift may be obtained by decreasing the aspect ratio. For example, for ellipsoids having a thickness ratio of 0.10, a decrease in the aspect ratio from ∞ to 2 causes the critical Mach number to increase from 0.827 to 0.857 (a Mach number increase of 0.03). For a thickness ratio of 0.20, a decrease in the aspect ratio from ∞ to 2 causes the critical Mach number to increase from 0.741 to 0.783 (about 0.04). Although ellipsoids having greater thickness ratio have lower critical Mach numbers, a decrease in the aspect ratio is slightly more effective in increasing the critical Mach numbers for ellipsoids of greater thickness ratio. Figure 7 indicates that only a large reduction in aspect ratio will cause a significant rise of the critical Mach number.

Comparison with test results on low aspect ratio wings.- Figure 6 of reference 1 shows the minimum drag coefficient $\left(C_D \text{ for zero lift}\right)$ plotted against the Mach number for wings having an NACA 0012 section and various aspect ratios. The critical Mach number for any aspect ratio may be estimated roughly as the Mach number for which the drag coefficient first begins to rise. The rough estimate of the critical Mach numbers obtainable by this consideration is not sufficiently accurate to warrant comparison of the numerical values with the numerical values of the critical Mach number obtained in the present paper for thin ellipsoids. Comparison of the numerical results is, moreover, not warranted inasmuch as the wings of reference 1 did not have an elliptic section and furthermore had a rectangular plan form. A qualitative comparison may be made, however, between the results of the present paper and those of reference 1. The increase in critical Mach number with decrease in aspect ratio indicated in figures 1, 2, 3, and 7 of the present paper is considered sufficiently large to explain the corresponding effect indicated in figure 6, reference 1.

It is mentioned in reference 1 that the Mach number for a significant rise in the drag coefficient is approximately 0.1 higher for an aspect ratio of 2 than for an infinite aspect ratio. This value is appreciably higher than the increase in critical Mach number due to a decrease in the aspect ratio. Since, for low-aspect-ratio wings, the drag coefficient increases only gradually after the critical Mach number is reached, the critical Mach number for a wing having low aspect ratio does not indicate so critical a change in the flow phenomena as the critical Mach number for a wing having high aspect ratio. It is thought that the smaller rate of increase of the drag coefficient for wings having low aspect ratio is due to the fact that, at the critical Mach number, the rate of increase with Mach number of the incremental velocity is less than for high aspect ratios, as may be seen from figures 1 to 3.

CONCLUSIONS

A study by the Prandtl-Glauert method of compressibility effects and critical Mach number for ellipsoids of various aspect ratios and thickness ratios indicated the following conclusions:

1. The flow about the unyawed ellipsoid is analogous to that about the infinite unyawed cylinder in that sonic velocity is reached simultaneously along a line that extends across the entire span of the body and is normal to the stream direction.

2. The critical Mach number for a thin ellipsoid may be predicted with good accuracy by means of the Prandtl-Glauert method, and the accuracy increases with decrease in aspect ratio.

3. The compressibility effect on the flow about an ellipsoid decreases as the aspect ratio decreases.

4. The three-dimensional relief for ellipsoids is essentially independent of the thickness ratio, for thickness ratios from 0.10 to 0.20.

5. For ellipsoids of thickness ratio 0.20, the critical Mach number increases by about 0.04 when the aspect ratio is changed from ∞ to 2; for ellipsoids of thickness ratio 0.10 the increase is 0.03.

6. The calculated increases in critical Mach number are sufficiently large to explain the experimentally observed increases in the Mach number at which the drag first begins to rise.

7. The experimentally indicated reduced rate of drag rise for low-aspect-ratio wings at zero lift as compared with that for wings having infinite aspect ratio may be explained qualitatively on the basis of the results obtained for the three-dimensional relief for ellipsoids.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
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APPENDIX

THE PRANDTL-GLAUERT METHOD FOR THREE-DIMENSIONAL FLOW

A derivation of the Prandtl-Glauert method for three-dimensional flow.- A brief derivation of a form of the Prandtl-Glauert method correct for three dimensions may be given as follows: A first-order approximation to the subsonic compressible flow about a thin body B, the surface of which has the equation

$$S(x, y, z) = 0$$

may be obtained by finding a solution of the linearized differential equation for the potential φ of the incremental velocities,

$$\beta^2\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \tag{A1}$$

where the x-axis is in the stream direction and the incremental velocities φ_x, φ_y, and φ_z are small compared with the stream velocity U. At all points on the surface of B, the potential φ must satisfy the boundary condition

$$(U + \varphi_x)\, S_x + \varphi_y S_y + \varphi_z S_z = 0 \tag{A2}$$

which states that the flow is tangential to B. Since B is assumed thin, S_x is small compared with S_y and S_z; consequently the second-order term $\varphi_x S_x$ may be neglected, and the boundary condition becomes

$$US_x + \varphi_y S_y + \varphi_z S_z = 0$$

In order to solve the boundary-value problem given by equations (A1) and (A2) in terms of incompressible flow, the following transformation of variables is used:

$$\left.\begin{aligned} x' &= \frac{x}{\beta} \\ \varphi' &= \beta\varphi \end{aligned}\right\} \tag{A3}$$

With this transformation equations (A1) and (A2) become, respectively,

$$\varphi'_{x'x'} + \varphi'_{yy} + \varphi'_{zz} = 0 \tag{A4}$$

$$US_{x'} + \varphi'_y S_y + \varphi'_z S_z = 0 \tag{A5}$$

Equations (A4) and (A5) are, respectively, the differential equation and boundary condition for the potential φ' of the incremental velocities of an incompressible flow with free-stream velocity U, in the x', y, z space, about a thin body B', the surface of which has the equation

$$S(\beta x', y, z) = 0$$

The incremental velocities in the compressible flow are thus given by

$$\begin{aligned} u &= \varphi_x = \frac{1}{\beta^2}\varphi'_{x'} = \frac{1}{\beta^2}u' \\ v &= \varphi_y = \frac{1}{\beta}\varphi'_y = \frac{1}{\beta}v' \\ w &= \varphi_z = \frac{1}{\beta}\varphi'_z = \frac{1}{\beta}w' \end{aligned} \tag{A6}$$

where u, v, and w and u', v', and w' are the incremental velocities at corresponding points in the compressible flow about B and the incompressible flow about B', respectively.

The foregoing analysis establishes the Prandtl-Glauert method for three-dimensional flow in the following form: The incremental velocities at a point P of a three-dimensional compressible flow field about a thin body B may be obtained in three steps:

(1) The x-coordinates of all points of B are increased by the factor $1/\beta$, where

$$\beta = \sqrt{1 - M^2}$$

and where the x-axis is in the stream direction. This transformation takes B into a stretched body B'.

(2) The incremental velocities u', v', w', in the direction of the x-, y-, and z-axes, respectively, at the point P' in the flow field of B' corresponding to the point P in the flow field of B

are calculated as though B' were in an incompressible flow having the same free-stream velocity as the original compressible flow.

(3) The values u, v, and w of the incremental velocities at the point P in the compressible flow field of the original unstretched body are then found by the following equations:

$$u = \frac{1}{\beta^2} u'$$

$$v = \frac{1}{\beta} v'$$

$$w = \frac{1}{\beta} w'$$

Thus far it has been shown that, through the transformation given in this paper, a compressible flow that satisfies the boundary conditions for the body B is transformed into an incompressible flow satisfying the boundary conditions for the stretched body B'. It can be shown further that the stream lines of the compressible flow about B are transformed into stream lines of the incompressible flow about B'. Because of this fact, the method has been referred to in the literature (for example, reference 4) as the "streamline-analogy" method.

The proof is obtained simply by applying the transformations (A3) and (A6) to the equations for the streamlines of the incompressible flow about the stretched body B'

$$\frac{dx'}{U} = \frac{dy'}{v'} = \frac{dz'}{w'} \tag{A7}$$

The application of the transformations results in the equation of the streamlines for the compressible flow about the body B

$$\frac{dx}{U} = \frac{dy}{v} = \frac{dz}{w} \tag{A8}$$

<u>Failure for three-dimensional flow problems of the commonly stated forms of the Prandtl-Glauert method</u>.- According to the form of the Prandtl-Glauert method given by Prandtl (reference 5) and Von Kármán (reference 6), the incremental velocities for a compressible flow about a thin body B are the same as the incremental velocities of corresponding points for incompressible flow having the same free-stream velocity about a body obtained by expanding B in the directions normal to the free-stream direction by the factor $1/\beta$. That is, for bodies of revolution, or two-dimensional bodies,

$$\frac{u}{U}(\epsilon,\ M) = \frac{u}{U}\left(\frac{1}{\beta}\epsilon,\ 0\right)$$

According to Göthert's method, however,

$$\frac{u}{U}(\epsilon,\ M) = \frac{1}{\beta^2}\frac{u}{U}(\beta\epsilon,\ 0) \tag{A9}$$

Thus, Prandtl's and Von Kármán's method is valid only if

$$\frac{u}{U}\left(\frac{1}{\beta}\epsilon,\ 0\right) = \frac{1}{\beta^2}\frac{u}{U}(\beta\epsilon,\ 0)$$

that is, if and only if the incremental velocity for incompressible flow about the bodies under consideration is proportional to the thickness ratio. This relation is approximately valid for thin two-dimensional bodies, so that the method of Prandtl and Von Kármán may be expected to be valid for two-dimensional flows. The relation is not true in general for three-dimensional bodies; for example, for a very thin body of revolution the incremental velocity is more nearly proportional to the square of the thickness ratio than to the first power.

Von Kármán approaches the problem by making the transformation

$$y' = \beta y$$

$$z' = \beta z$$

$$\varphi' = \varphi$$

Under this transformation the linearized equation of compressible flow goes into Laplace's equation; however, the transformed boundary condition is not satisfied on the surface of the transformed (contracted) body but on the surface of an expanded body. Thus, the boundary condition is not satisfied on the boundary but at points near the boundary. This procedure is applicable to two-dimensional problems (as, for example, in the thin-wing theory, reference 14), because the velocity increments induced by the equivalent line distribution of singularities vary only slowly in the neighborhood of the line of singularities. For a body of revolution, however, the velocity increments induced by a line of singularities go to infinity at the line of singularities; for such bodies, accordingly, the location of the point at which the boundary condition is satisfied is important.

According to Goldstein and Young (reference 7), "in compressible flow the pressure increase at any point of the body is $1/\beta$ times the pressure increase in incompressible flow at the same point." That is,

$$\frac{u}{U}(\epsilon, M) = \frac{1}{\beta}\frac{u}{U}(\epsilon, 0)$$

Comparison of this relation with equation (A9) shows that the Goldstein-Young method is also valid for two-dimensional problems but gives an incorrect result for three-dimensional problems.

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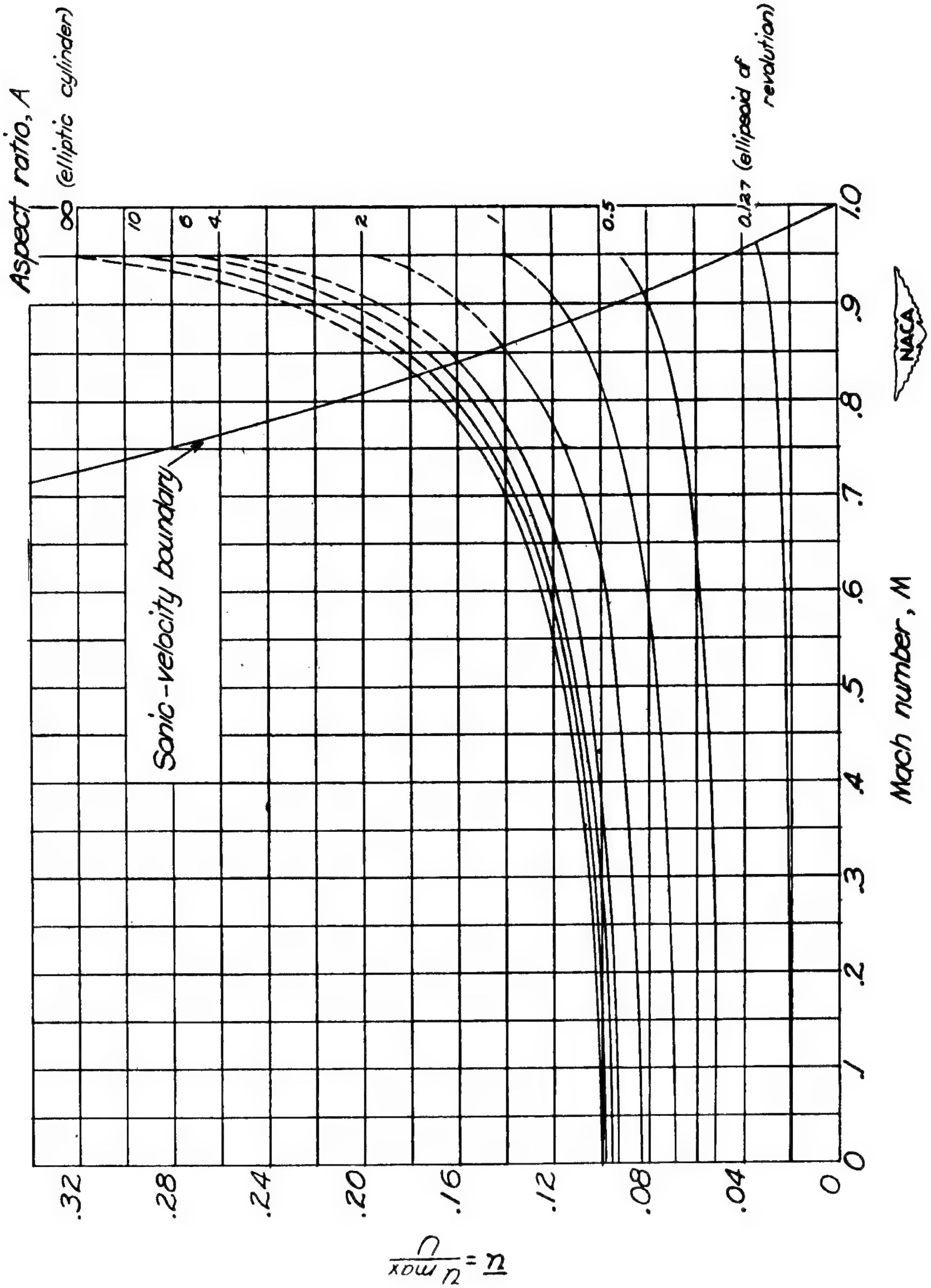

Figure 1.- Variation with Mach number of maximum incremental velocity for ellipsoids having various aspect ratios. Thickness ratio, 0.10.

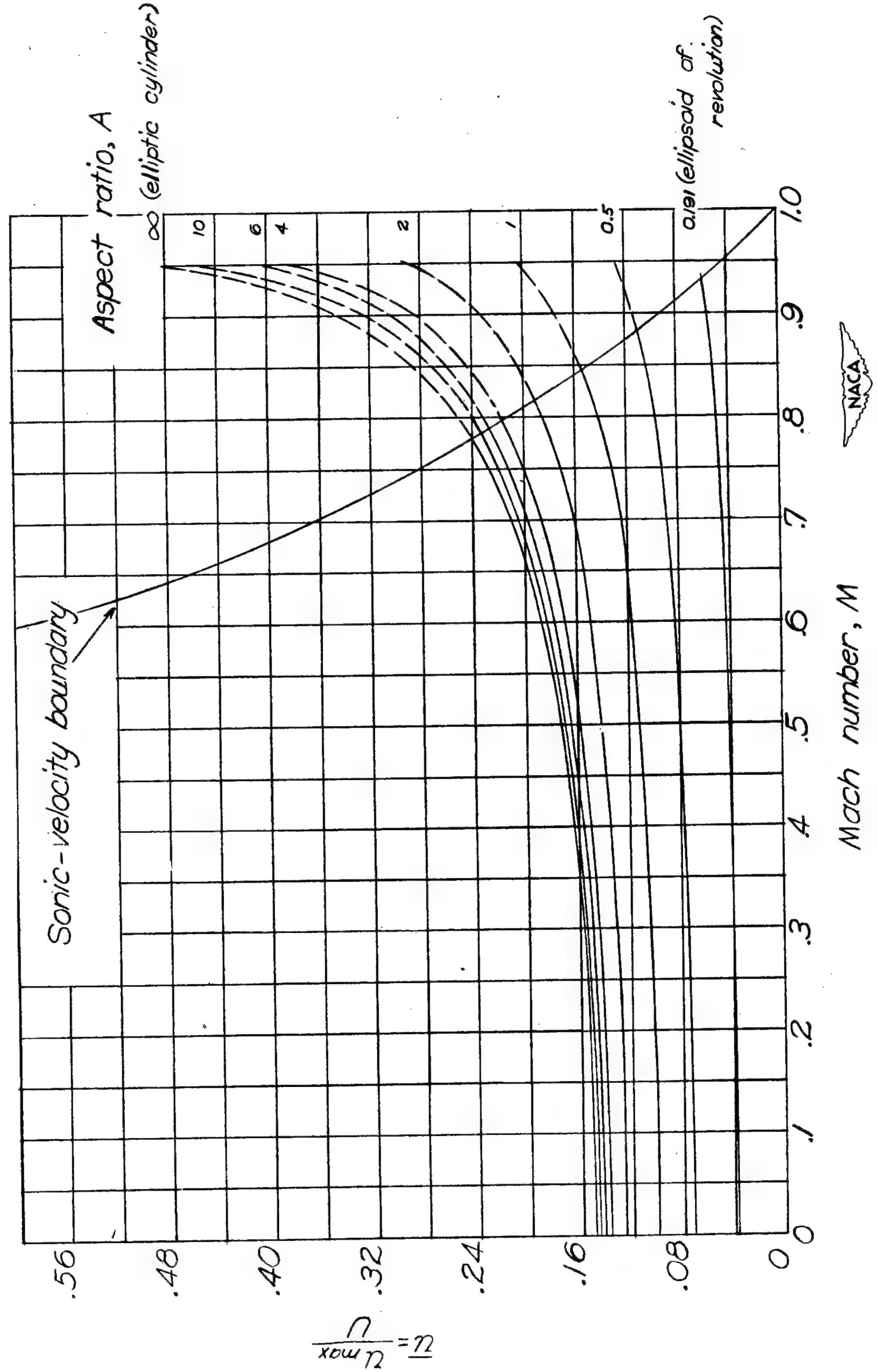

Figure 2.- Variation with Mach number of maximum incremental velocity for ellipsoids having various aspect ratios. Thickness ratio, 0.15.

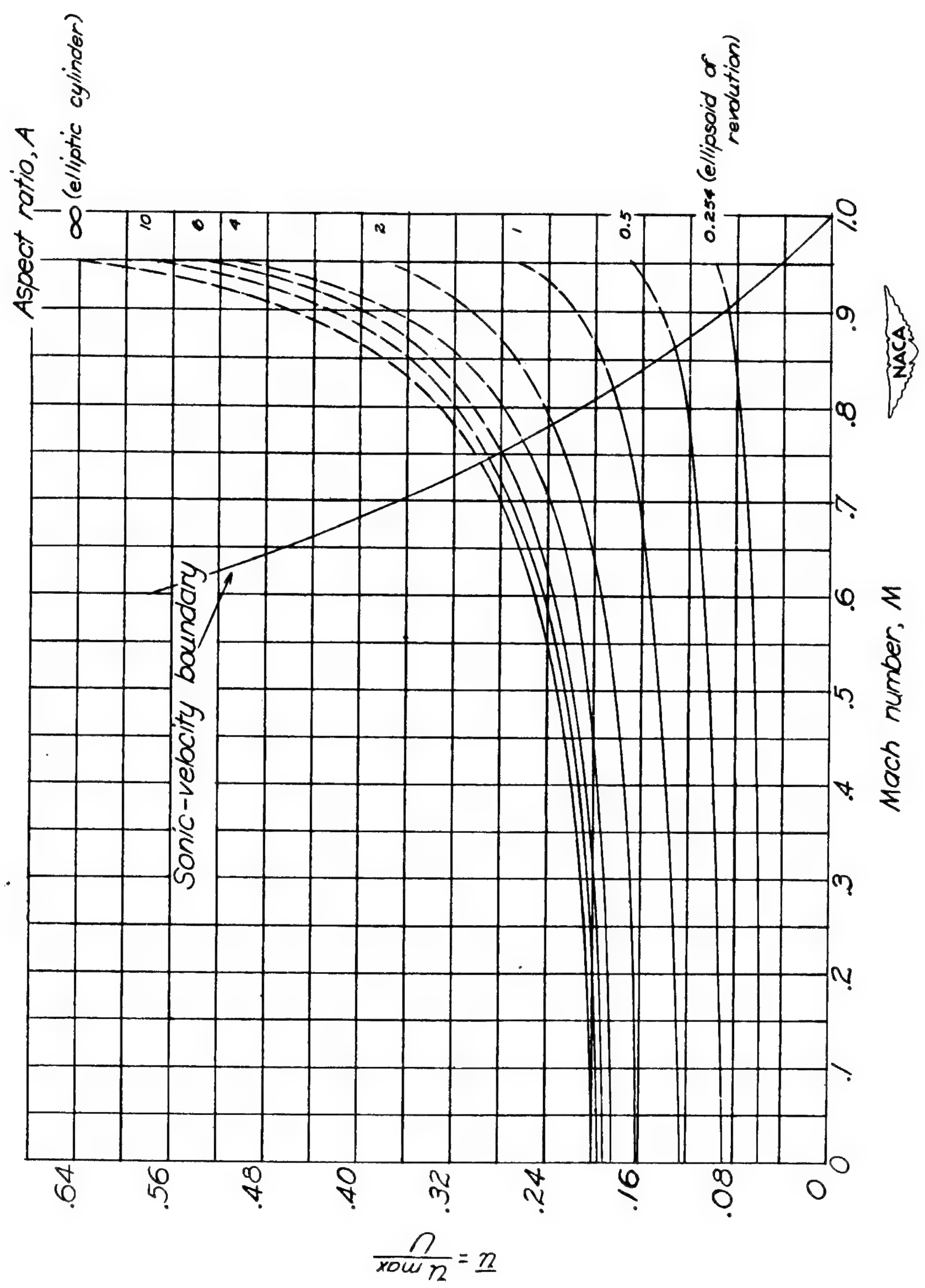

Figure 3.- Variation with Mach number of maximum incremental velocity for ellipsoids having various aspect ratios. Thickness ratio, 0.20.

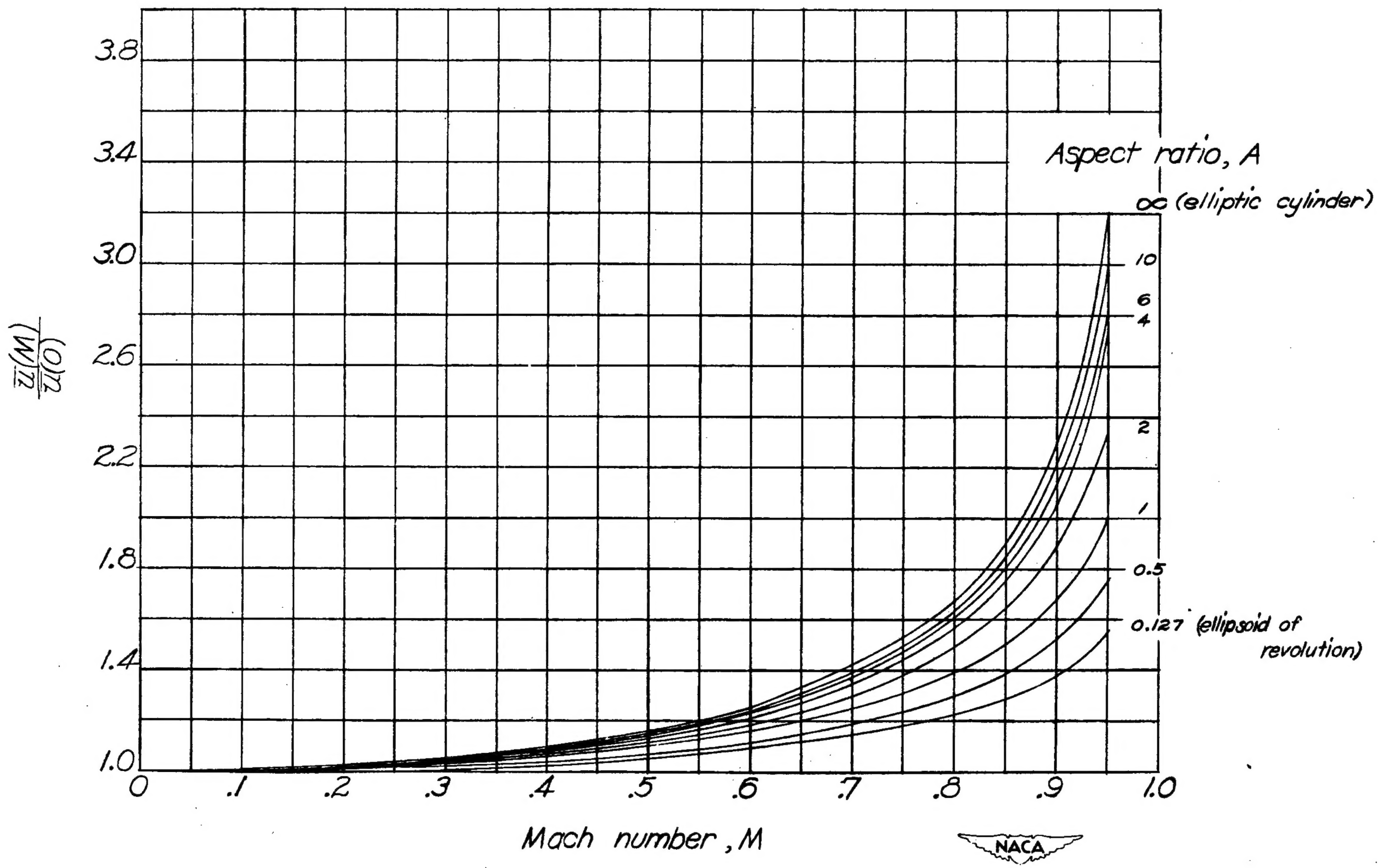

Figure 4.- Variation of compressibility effect with Mach number for ellipsoids having various aspect ratios. Thickness ratio, 0.10.

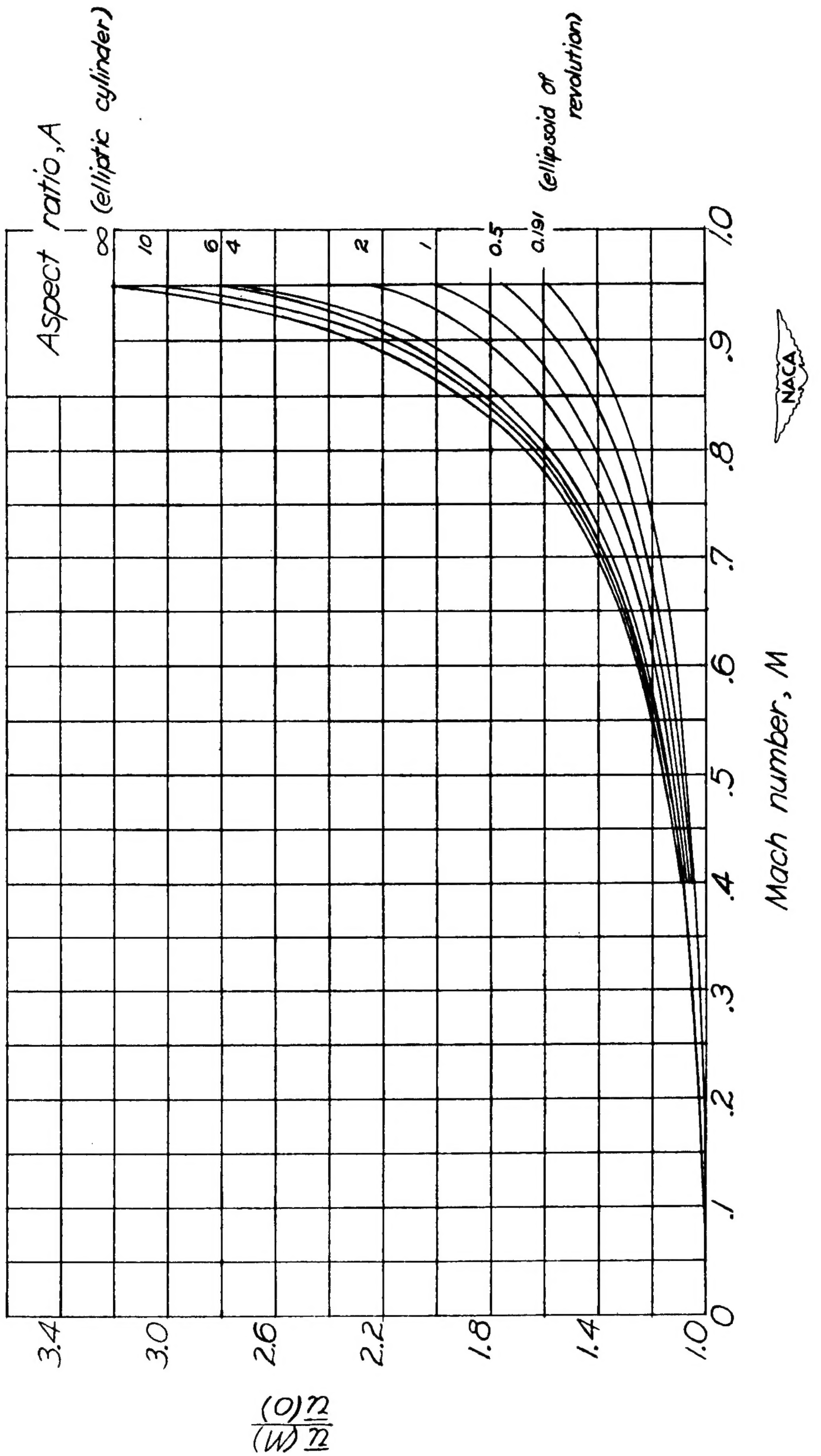

Figure 5.- Variation of compressibility effect with Mach number for ellipsoids having various aspect ratios. Thickness ratio, 0.15.

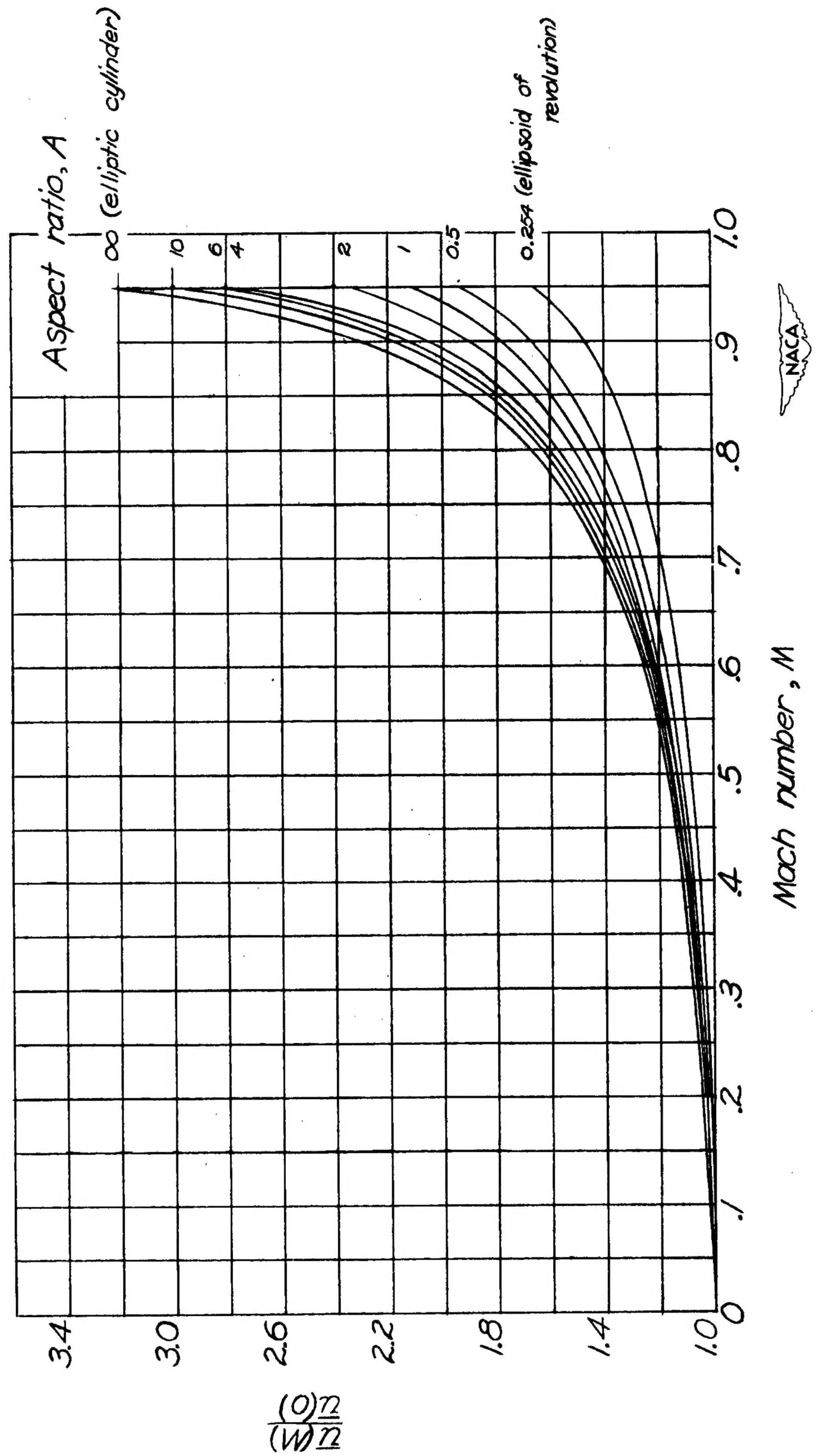

Figure 6.- Variation of compressibility effect with Mach number for ellipsoids having various aspect ratios. Thickness ratio, 0.20.

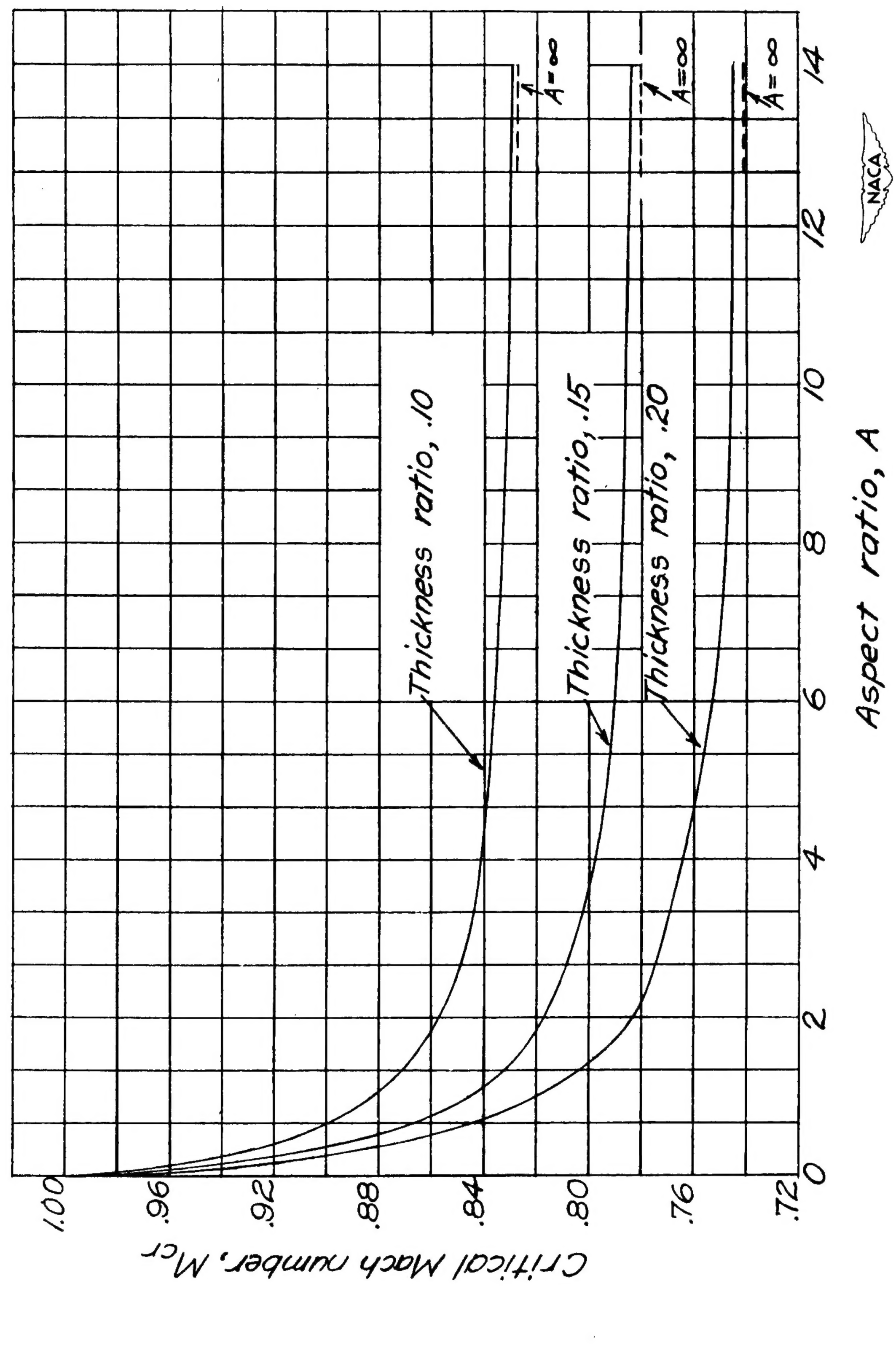

Figure 7.- Variation of critical Mach number with aspect ratio for ellipsoids having thickness ratios of 0.10, 0.15 and 0.20.